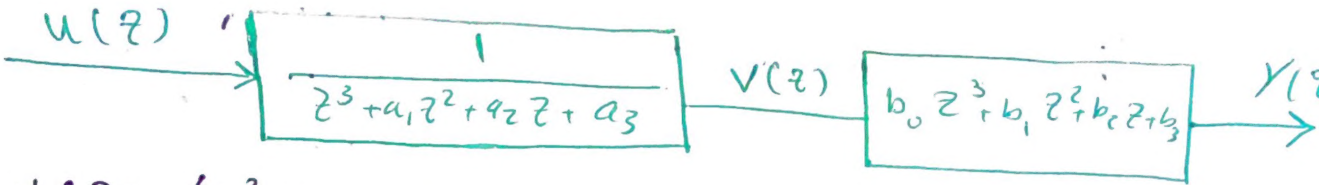
Digital

18/4/2016 السادس د. عرضة محاضرة 8

Quiz

$$T.F. = \frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

3rd order System




$$U(z) = (z^3 + a_1 z^2 + a_2 z + a_3) V(z)$$

$$\xrightarrow{z^{-1}T} u(k) = v(k+3) + a_1 v(k+2) + a_2 v(k+1) + a_3 v(k)$$

$x_1(k) = v(k)$

$x_2(k) = v(k+1) = x_1(k+1)$

$x_3(k) = v(k+2) = x_2(k+1)$

$x_3(k+1) = v(k+3)$

$\Rightarrow$

$x_3(k+1) = v(k+3)$

$v(k+3) = -a_3 v(k) - a_2 v(k+1) - a_1 v(k+2) + u(k)$

$x_3(k+1) = -a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k)$

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} X(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (\qquad\qquad) X(k) + D\, u(t)$$

$$\frac{Y(z)}{V(z)} = b_0 z^3 + b_1 z^2 + b_2 z + b_3$$

$$Y(z) = (b_0 z^3 + b_1 z^2 + b_2 z + b_3) V(z)$$

$$\Downarrow z^{-1}\ T$$

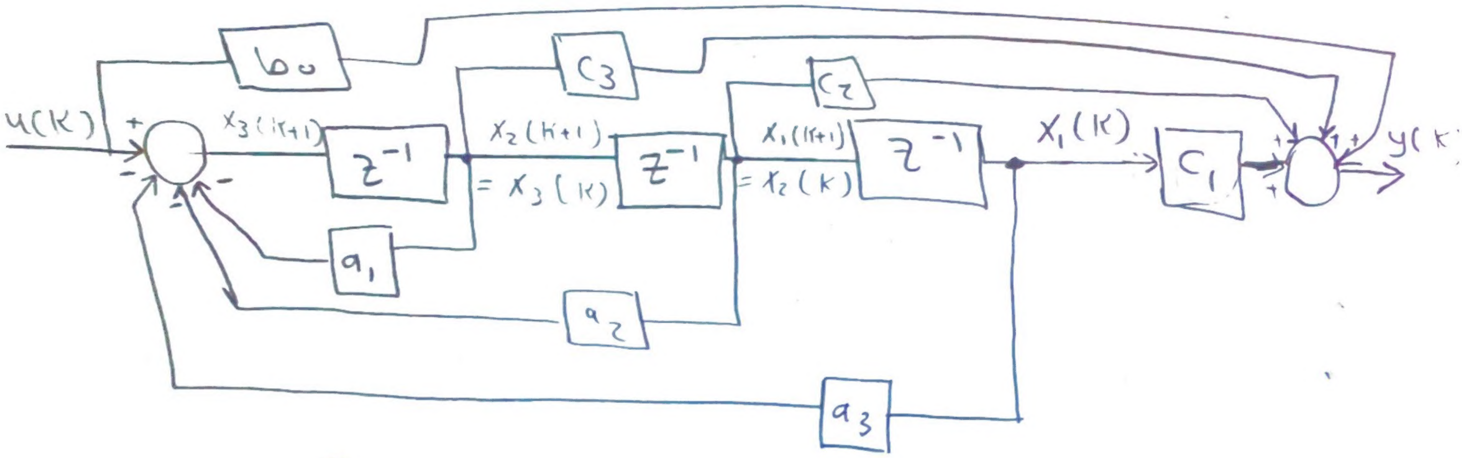
$$y(k) = b_0 \underbrace{v(k+3)}_{x_3(k+1)} + b_1 \underbrace{v(k+2)}_{x_3(k)} + b_2 \underbrace{v(k+1)}_{x_2(k)} + b_3 \underbrace{v(k)}_{x_1(k)}$$

$$y(K) = b_0\left(-a_3 X_1(K) - a_2 X_2(K) - a_1 X_3(K) + u(K)\right) + b_1 X_3(K) + b_2 X_2(K) + b_3 X_1(K)$$

$$y(K) = (b_3 - b_0 a_3)\, X_1(K) + (b_2 - b_0 a_0)\, X_2(K) + (b_1 - b_0 a_1)\, X_3(K) + b_0 u(K)$$

$D_{1\times 1}$

$$y(K) = \begin{pmatrix} \underbrace{b_3 - b_0 a_3}_{c_1} & \underbrace{b_2 - b_0 a_2}_{c_2} & \underbrace{b_1 - b_0 a_1}_{c_3} \end{pmatrix} \begin{pmatrix} X_1(K) \\ X_2(K) \\ X_3(K) \end{pmatrix} + \overbrace{b_0}^{D_{1\times 1}} u(K)$$

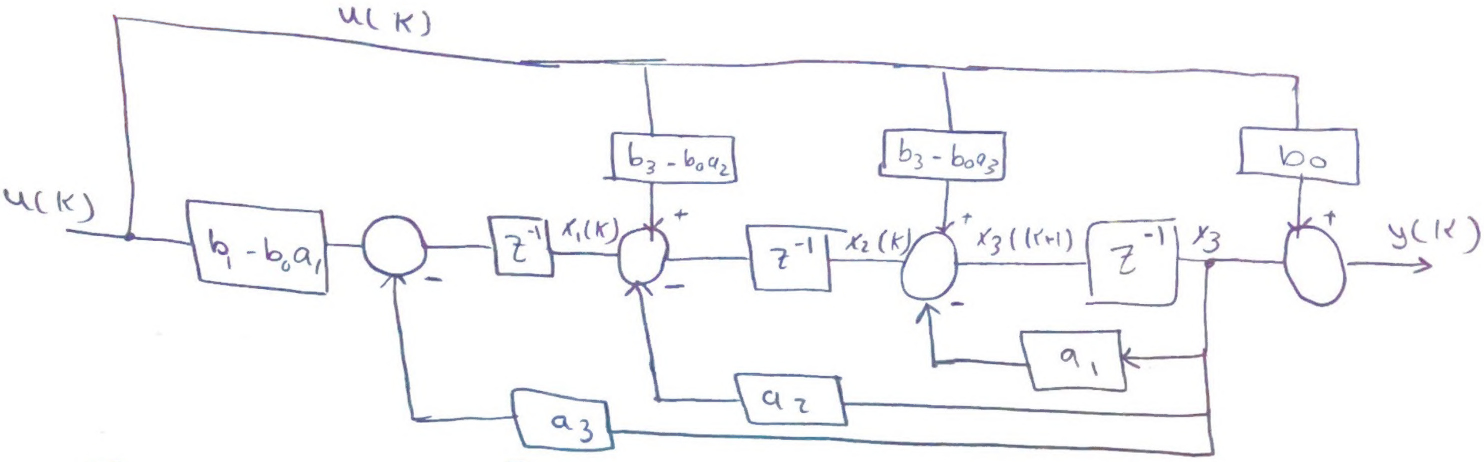



Observable form:

$A_o = A_c^T$ ; $B_o = C_c^T$ ; $C_o = B_c^T$ ; $D_o = D_c = b_o$

$$X(K+1) = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} X(K) + \begin{pmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{pmatrix} u(K)$$

$$y(K) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} X(K) + b_0 u(K)$$




Ex: $$T.F. = \frac{z^3+2z+3}{z^3+4z^2+2z-1}$$

Controllable Form:-

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 4 & 0 & -4 \end{pmatrix} x(k) + u(k)$$

$3-1(-1)$ $\quad 2-1(2)$ $\quad 0-1(4)$

observable form

$$x(k+1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(k) + u(k)$$

Diagonal form

$$T.F. = \frac{2z^3+z+1}{z(z+0.5)(z-0.5)} \Leftarrow z(z^2-0.25)$$

$$\begin{array}{r|l} & 2 \\ \hline z^3-0.25z & 2z^3 + z + 1 \\ & 2z^3 - 0.5z \\ \hline & 1.5z+1 \end{array}$$

$$T.F. = 2 + \frac{1.5z+1}{z^3 - 0.25z}$$

$$= 2 + \frac{1.5z+1}{z(z+0.5)(z-0.5)} \Downarrow \text{using P.F.}$$

$$= \underbrace{2}_{D_{1\times1}} + \frac{A_1}{z} + \frac{A_2}{z+0.5} + \frac{A_3}{z-0.5}$$

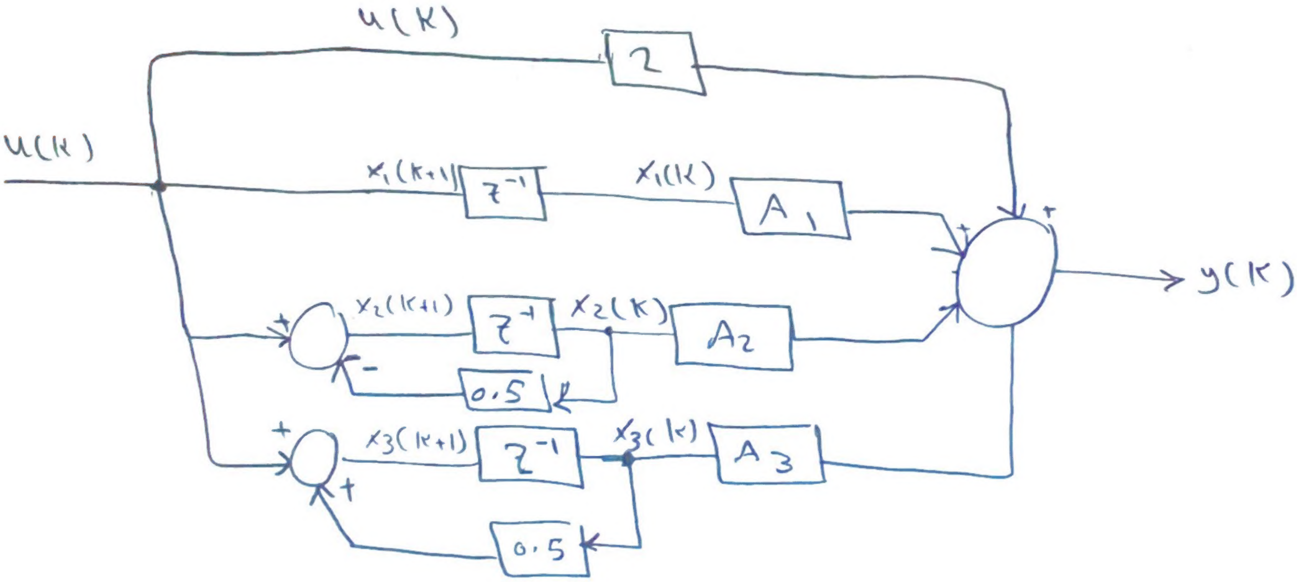
$$A_1 = \frac{1}{(0.5)(-0.5)}$$

$$A_2 = \frac{1.5(-0.5)+1}{(-0.5)(-1)}$$

$$A_3 = \frac{(1.5)(0.5)+1}{(0.5)(1)}$$

$$X(k+1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} X(k) + 2u(k)$$




⟹ State Space Analysis

# State Space Analysis :-

* given $X(K+1) = A\,x(K) + B\,u(K)$ $\quad$ A, B, C, D
$y(K) = C\,X(K) + D\,u(K)$ $\quad$ are given

## ① T.F.

$$X(K+1) = A\,x(K) + B\,u(K) \xrightarrow[(Z.I.C)]{Z.T} Z\,X(Z) = A\,X(Z) + B\,u(Z)$$

$$Z\,X(Z) - A\,X(Z) = B\,u(Z)$$

$$(ZI - A)\,X(Z) = B\,u(Z)$$

$$X(Z) = (ZI - A)^{-1} B\,u(Z) \quad ①$$

$$y(K) = C\,X(K) + D\,u(K) \quad \downarrow Z.T$$

$$Y(Z) = C\,X(Z) + D\,u(Z) \quad ②$$

$$Y(Z) = C(ZI - A)^{-1} B\,u(Z) + D\,u(Z)$$

$$T.F. = \frac{Y(Z)}{u(Z)} = C\,(ZI - A)^{-1} B + D$$

## ② Ch. eqn

$$|ZI - A| = 0 \quad \Big\| \quad T.F. = \frac{\text{num}}{\text{den}}$$

den $= 0$

## ③ check the system controllability & observability.

Controllability :-

The system is contrallable if for any change of an external input, produce change in internal states of the system.

$\Longrightarrow$ observability

observability :-

the system is observable if we can determine or estimate The States values from the relation between i/p and o/p or by the history information for the o/p and i/p

# Controllability Matrix ($M_c$)

$M_c = (B \quad AB \quad A^2B \; ---- \; A^{n-1}B)$ n → system order

for 2nd order

$M_c = (B \quad AB)$

for 3rd order

$M_c = (B \quad AB \quad A^2B)$

if $|M_c| \neq 0$, the system is controllable

# Observability Matrix ($M_o$)

$$M_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

for 2nd order

$$M_o = \begin{pmatrix} C \\ CA \end{pmatrix}$$

for 3rd order

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

⟹ $\boxed{4}$ System Response

## [4] The System Response:

if $x(0) = 0$ (Z.I.C)

① $T.F. = \dfrac{Y(z)}{U(z)} = C(zI-A)^{-1}B + D$

$$Y(z) = [T.F.] \cdot U(z) \leftarrow \text{given}$$

$$\downarrow$$

$$z^{-1}.T \Rightarrow y(k) = \checkmark$$

if $x(0) \neq 0$ (there is a value for initial states)

$$X(k+1) = AX(k) + Bu(k)$$

$$\downarrow z.T$$

$$zX(z) - zX(0) = AX(z) + Bu(z)$$

$$zX(z) - AX(z) = zX(0) + Bu(z)$$

$$(zI - A)X(z) = zX(0) + Bu(z)$$

$$\boxed{X(z) = (zI-A)^{-1} zX(0) + (zI-A)^{-1} B\, u(z)}$$

Let $\phi(z) = (zI-A)^{-1}$

$$\boxed{X(z) = \phi(z)\, zX(0) + \phi(z) B u(z)}$$

$$\Downarrow z^{-1}.T$$

$X(k)$

System Response: $y(k) = CX(k) + Du(k)$

EX:

$$x(k+1) = \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & -0.6321 \end{pmatrix} x(k)$$

① Find T.F.

② Check Stability

$$T.F. = C\,(zI-A)^{-1} B + \underbrace{\cancel{D}}_{=0}$$

$$(zI-A) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z & -0.6321 \\ 1 & z-1 \end{pmatrix} \Longleftarrow \text{get det.}$$

$$(zI-A)^{-1} = \frac{1}{z(z-1)+0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix}$$

$$T.F. = C\,(zI-A)^{-1} B$$

$$= \begin{pmatrix} 1 & -0.6321 \end{pmatrix} \frac{1}{z^2 - z + 0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{z^2 - z + 0.6321} \begin{pmatrix} 1 & -0.6321 \end{pmatrix} \begin{pmatrix} z-1+0.6321 \\ -1+z \end{pmatrix}$$

$$T.F. = \frac{z-1+0.6321+0.6321-0.6321z}{z^2 - z + 0.6321}$$

$$T.F. = \frac{0.3679z + 0.2642}{z^2 - z + 0.6321}$$

ch. eqn $z^2 - z + 0.6321 = 0$

$z_{1,2} = 0.5 \pm j\, 0.618$
poles

$|z_{1,2}| = 0.795 < 1 \Longrightarrow$ system Stable

Ex:

$$X(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} X(k)$$

if $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find unit Step response.

$$X(z) = \phi(z)\, z\, x(0) + \phi(z)\, B u(z)$$

$$\phi(z) = (zI - A)^{-1}$$

$$(zI - A) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} z-0.8 & 0 \\ 0 & z-0.2 \end{pmatrix}$$

$$\phi(z) = (zI - A)^{-1} = \frac{1}{(z-0.8)(z-0.2)} \begin{pmatrix} z-0.2 & 0 \\ 0 & z-0.8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix}$$

$u(t) = 1$
$\Rightarrow u(k) = 1 \longrightarrow u(z)$

$$x(z) = \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} z \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z+0.2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\frac{z}{z+}\right)$$

$$X(z) = \begin{pmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \frac{1}{z-0.8} \\ \frac{1}{z-0.2} \end{pmatrix} \left(\frac{z}{z-1}\right)$$

$$X(z) = \begin{pmatrix} \dfrac{z}{z-0.8} \\[2ex] \dfrac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{(z-0.8)(z-1)} \\[2ex] \dfrac{1}{(z-0.2)(z-1)} \end{pmatrix} z$$

$$= \begin{pmatrix} \dfrac{z}{z-0.8} \\[2ex] \dfrac{z}{z-0.2} \end{pmatrix} + \begin{pmatrix} \dfrac{-5}{z-0.8} + \dfrac{5}{z-1} \\[2ex] \dfrac{-1.25}{z-0.5} + \dfrac{1.25}{z-1} \end{pmatrix} z$$

$$z^{-1}.T \Rightarrow \quad x(k) = \begin{pmatrix} (0.8)^k \\ (0.2)^k \end{pmatrix} + \begin{pmatrix} -5\,(0.8)^k + 5 \\ -1.25\,(0.2)^k + 1.25 \end{pmatrix}$$

$$x(k) = \begin{pmatrix} (0.8)^k - 5\,(0.8)^k + 5 \\ (0.2)^k - 1.25\,(0.2)^k + 1.25 \end{pmatrix}$$

$$= \begin{pmatrix} -4\,(0.8)^k + 5 \\ -0.25\,(0.2)^k + 1.25 \end{pmatrix}$$

The unit step response $y(k)$ is

$$y(k) = \begin{pmatrix} 1 & & 1 \end{pmatrix} \begin{pmatrix} -4\,(0.8)^k + 5 \\ -0.25\,(0.2)^k + 1.25 \end{pmatrix}$$

$$= -4\,(0.8)^k - 0.25\,(0.2)^k + 6.25$$

$\Longrightarrow$ Another example

Ex:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$$

check system's controllability and observability

$$M_c = \begin{pmatrix} B & AB & A^2B \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

$$A^2B = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}}_{A} * \underbrace{\begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}}_{AB} = \begin{pmatrix} -2 \\ 7 \\ -20 \end{pmatrix}$$

$$M_c = \begin{pmatrix} +0 & 3 & -2 \\ -1 & 11 & 7 \\ +3 & -5 & -20 \end{pmatrix} \Rightarrow |M_c| = -1 \begin{vmatrix} 3 & -2 \\ -5 & -20 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 11 & 7 \end{vmatrix}$$

$$|M_c| = 70 + 129 = 199 \neq 0$$

$\Rightarrow$ System is Controllable

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}; \quad CA = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -1 \end{pmatrix}$$

$$CA^2 = CA \cdot A = \begin{pmatrix} -1 & -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -6 & -12 \end{pmatrix}$$

$\Rightarrow$ get $|M_o|$

$$M_o = \begin{array}{c} + \\ - \\ + \end{array} \begin{pmatrix} 1 & 0 & 2 \\ -1 & -4 & -1 \\ 0 & -6 & -12 \end{pmatrix}$$

$$= \begin{vmatrix} -4 & -1 \\ -6 & -12 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ -6 & -12 \end{vmatrix} = 54 \neq 0$$

System is observable